First Order Linear (F.O.L.) ODEs

A DE is of the form

$$\frac{dy}{dx} + Py = Q \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (i)$$

where P,Q are the functions of **x or constants** is called linear differential equation.

Integrating factor (I.F.): An equation is readily integrable when the equation is multiplied by a factor $e^{\int Pdx}$ then such a factor is called **I.F.**

Construction : for $\frac{dy}{dx} + Py = Q \ldots..(i)$

where P,Q are the functions of x or constants
To solve this problem multiply both sides of equation (i) by $e^{\int Pdx}$
Then it becomes

$$e^{\int Pdx}\frac{dy}{dx} + Py\, e^{\int Pdx} = Q\, e^{\int Pdx}$$

$$\Rightarrow \frac{d}{dx}\left[y\, e^{\int Pdx}\right] = Q\, e^{\int Pdx}$$

Integrating both sides with respect to x (w.r.t), we get

$$y\, e^{\int Pdx} = \int Q\, e^{\int Pdx}\, dx + c$$

which is the required solution.

Construction : for $\frac{dx}{dy} + Px = Q \ldots.(i)$

where P,Q are the functions of y or constants
To solve this problem multiply both sides of equation (i) by $e^{\int Pdy}$
Then it becomes

$$e^{\int Pdy}\frac{dx}{dy} + Py\, e^{\int Pdy} = Q\, e^{\int Pdy}$$

$$\Rightarrow \frac{d}{dy}\left[y\, e^{\int Pdy}\right] = Q\, e^{\int Pdy}$$

Integrating both sides with respect to y (w.r.t) , we get

$$x\, e^{\int Pdy} = \int Q\, e^{\int Pdy}\, dy + c$$

which is the required solution.

Solve: $(1 - x^2)\frac{dy}{dx} - xy = 1$

The DE can be written in the form

$$\frac{dy}{dx} - \frac{x}{(1 - x^2)}y = \frac{1}{(1 - x^2)}$$

Where $P = -\frac{x}{(1-x^2)}$ and $Q = \frac{1}{(1-x^2)}$

$IF = e^{\int Pdx} \Rightarrow e^{\int -\frac{x}{(1-x^2)}dx}$

Let $z = 1 - x^2$ then $dz = -2x\, dx$

$$\Rightarrow -x\, dx = \frac{dz}{2}$$

So

$$e^{\int \frac{dz}{2z}} = e^{\frac{1}{2}\int \frac{1}{z}dz} = e^{\frac{1}{2}\ln z} = e^{\ln \sqrt{z}} = \sqrt{z} = \sqrt{1 - x^2}$$

Hence the solution is

$$y\, e^{\int Pdx} = \int Q\, e^{\int Pdx}\, dx + c$$

$$\Rightarrow y\sqrt{1 - x^2} = \int \frac{1}{(1 - x^2)}\sqrt{1 - x^2}\, dx + c$$

$$\Rightarrow y\sqrt{1 - x^2} = \int \frac{1}{\sqrt{1 - x^2}}\, dx + c$$

$$\Rightarrow y\sqrt{1 - x^2} = \sin^{-1} x + c$$

Exercises :

(i) $x\frac{dy}{dx} + 2y = x^4$

(ii) $\frac{dy}{dx} = \frac{y}{x} + 2x^2$